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A NOTE ABOUT THE SOLUTION OF MATRIX SYLVESTER EQUATION

F.A. ALIEV[1], V.B. LARIN[2]

Abstract. Algorithms for solving the matrix Sylvester equation are reviewed. In the first algorithm the specific form of the matrix Sylvester equation is used. The possibility of using the procedures of symbolic calculation for increasing the accuracy of this algorithm is discussed. In the second algorithm the techniques of linear matrix inequalities are used. The comparison of accuracy of these algorithms is discussed for some examples.

Keywords: matrix Sylvester equation, MATLAB package, Symbolic Math Toolbox, LMI method, SV Decomposition.

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1. Introduction

The problem of developing the algorithms for solving the Sylvester equation

$$AX - XB = C. \tag{1}$$

have attracted attention of researchers (see [2, 5, 8, 10, 12] and references therein). In (1), A, B are quadratic matrices of sizes $A \in C^{n\times n}$, $B \in C^{m\times m}$. In the iterative procedure for finding the solution of an asymmetric Riccati equation, considered in [1, 3, 8, 9], a problem arises (relation (2.1) [8]) for constructing the solution (1) supplemented by the following relation:

$$DX = G, \tag{2}$$

where D, G are matrices of appropriate dimensions.

Below the problem for finding the solution of system (1), (2) is considered. In this connection, it is proposed to use the algorithm [11] to construct the solution of (1), which allows one to transform the system (1), (2) into the system (10). In turn, this allows one to use the standard MATLAB package procedures to find the desired matrix X.

2. Equation (1)

In [11], a finite expression for the matrix satisfying the matrix equation (1) is given.

According to [11], the solution of this equation has the form

$$X = -\left(C_n + p_1 C_{n-1} + \ldots + p_{n-1} C_1\right)\left[P_A(B)\right]^{-1}, \tag{3}$$

where $P_A(t)$ is the characteristic polynomial of the matrix

$$P_A(t) = t^n + p_1 t^{n-1} + \ldots + p_{n-1} t + p_n, \tag{4}$$

[1]Institute of Applied Mathematics, Baku State University, Baku, Azerbaijan
[2]Institute of Mechanics Academy of Sciences of Ukraine, Ukraine
e-mail: f_aliev@yahoo.com

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and the matrices C_n are defined by

$$C_n = \sum_{k=1}^{n} A^{n-k} C B^{k-1}. \tag{5}$$

The expression (3) is a consequence of the more general relation obtained in [11], namely, if

$$\Pi(t) = t^n + \pi_1 t^{n-1} + \ldots + \pi_{n-1} t + \pi_n \tag{6}$$

is a certain polynomial, then the solution of equation (3) satisfies the following relation:

$$\Pi(A)X - X\Pi(B) = C_n + \pi_1 C_{n-1} + \ldots + \pi_{n-1} C. \tag{7}$$

The formula (3) can be obtained if $\Pi(t)$ in (8) is chosen to be equal to $P_A(t)$.
If in (7) we choose as the polynomial the characteristic polynomial of the matrix B:

$$P_b(t) = t^m + q_1 t^{m-1} + \cdots + q_{m-1} t + q_m, \tag{8}$$

then the solution of (1) will satisfy the following relation:

$$P_b(A)X = C_m + q_1 C_{m-1} + \cdots + q_{m-1} C. \tag{9}$$

The matrices C_ℓ $(\ell = 1 : m)$ from (9) are determined by relations analogous to (5).
Complementing (9) with the relation (2), we obtain the system of equations that determine the unknown matrix$X$:

$$\begin{bmatrix} P_b(A) \\ D \end{bmatrix} X = \begin{bmatrix} C_q \\ G \end{bmatrix}, C_q = C_m + q, C_{m-1} + \cdots + q_{m-1} C. \tag{10}$$

The various algorithms can be used for solving the system (10), in particular, the procedure "\" of the MATLAB package.

Note that for obtaining the solution in the more general case (when the system consists of two pairs of equations (1), (2) (the relations (3.2) [5])), it is possible to effectively use the algorithm [4] to find the solution of the system of periodic Sylvester equations.

3. Example 1

Let the matrices A, B, C, D, G from equations (1), (2) have the form:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}, C = \begin{bmatrix} -2 & -12 \\ 8 & 0 \\ 20 & 10 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, G = \begin{bmatrix} 6 & 6 \end{bmatrix}.$$

The corresponding polynomial is$P_b(A) = A^2 - 7A$. According to (10), the matrix C_q has the form:

$$C_q = \begin{bmatrix} 10 & 6 \\ 20 & 12 \\ 36 & 44 \end{bmatrix}.$$

The exact solution X of the system (1), (2) with this initial data has the form:

$$X = \begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 3 & 1 \end{bmatrix}.$$

As a result of solving the system (1), (2), a solution X_n is obtained, with the corresponding inaccuracy $er = norm(X - X_\Pi, \inf)$ being

$$er = 2,6 \cdot 10^{-15}.$$

Note that the above procedures allows for the implementation of calculations with arbitrary accuracy by using the MATLAB package Symbolic Math Toolbox.

Next, let us consider the algorithm for solving equations (1), (2), which uses the procedures of linear matrix inequalities (LMN), namely, the LMI toolbox of the MATLAB package [7].

4. Solving equations (1), (2) using the LMI toolbox of the MATLAB package

Let us consider the corresponding procedures.

As noted in [7] (relations (2.3), (2.4)) lead to the matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0, \tag{11}$$

where the matrices $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, $S(x)$ are linearly depend on x . This is equivalent to the following matrix inequalities:

$$R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0. \tag{12}$$

Let us now consider the following LMI:

$$\begin{bmatrix} Z & T \\ T^T & I \end{bmatrix} > 0, Z = Z^T, \;\; Z < \lambda I. \tag{13}$$

Here and below the superscript denotes transposition, I is the unit matrix of appropriate size, λ is scalar. Taking into account (11), (12), the relations (13) can be rewritten as

$$Z > TT^T, \;\; Z < \lambda I \textit{ or } \lambda I > TT^T. \tag{14}$$

The relations (14) allow us to consider the following standard LMI problem for eigenvalues (p. 2.2.2 [6]), namely, the minimization problem of λ when conditions (14) are satisfied.
The relations (13) can be generalized in the form of the following LMI system:

$$\begin{bmatrix} Z & T_i \\ T_i^T & I \end{bmatrix} > 0, i = 1, 2, \ldots, kZ = Z^T, \;\; Z < \lambda I, \tag{15}$$

which can be represented in the form similar to (14):

$$Z > T\,_i T_i^T, \, i = 1, 2, \ldots, k \; Z < \lambda\, I. \tag{16}$$

With respect to (16), one can also consider the standard LMI problem for eigenvalues and use the gevp.m procedure of the MATLAB package to solve it [7].

We use the above relations to find the solution of equations (1), (2). Suppose that in (15) the matrices T_1, T_2 have the form

$$T_1 = AX - XB - C, \tag{17}$$

$$T_2 = DX - G. \tag{18}$$

If necessary, the matrices D, G in (18) need to be supplemented with zero blocks so that the matrices $T_1T_1^T$, $T_2T_2^T$ would be of the same size (see Example 2).

Using the gevp.m procedure of the MATLAB package, we find the minimum value of λ (and the corresponding value X), in which the relations (16) are satisfied. Obviously, for a sufficiently small λ, the norm of the matrices T_i will also be enough small, i.e. $T \cong 0$ and, consequently, obtained as a result of using the procedure gevp.m, the value X can be considered as the solution of equations (1), (2) obtained with a certain accuracy. In connection with the fact that one has not minimized the norm of the matrices T_i , but the norm of the matrices $T_iT_i^T$, we can expect a decrease of the accuracy of the result of solving of the equations (1), (2).

5. Example 2

The initial data coincide with those in Example 1, except the values of the matrices D and G, which, in accordance with the remark made above, respectively to the relations (17), (18), are taken in the following form

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 6 & 6 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

As a result of using the procedure described in p.4, a matrix X_Π , which is the solution of (1), (2) and the values of the matrices T_1, T_2 , were obtained, with the following norms of corresponding inaccuracies:

$$\|T_1\|_\infty = 1,3 \cdot 10^{-9},$$
$$\|T_2\|_\infty = 2,9 \cdot 10^{-9},$$
$$\|X_\Pi - X\|_\infty = 7,2 \cdot 10^{-9}.$$

Thus, the accuracy of the obtained solution (1), (2) in this example is lower than the accuracy provided by the algorithm p.2.

6. Conclusion

The algorithms for solving the Sylvester equation by an additional linear equation is considered. The first algorithm uses the specific form of the Sylvester equation. It is noted that it is possible to use the procedures of symbolic computations in this algorithm in order to improve the accuracy of the result. The second algorithm uses procedures of the linear matrix inequalities. The accuracies of these algorithms are compared in several examples.

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Fikret Aliev, for a photograph and biography, see TWMS J. Pure Appl. Math., V.1, N.1, 2010, p.12

Vladimir Larin, for a photograph and biography, see TWMS J. Pure Appl. Math., V.2, N.1, 2011, p.160